3201. Completing the square in each, the graphs can be expressed, for some $p, q, r, s \in \mathbb{R}$, as

$$\begin{aligned} y &= (x-p)^2 + q, \\ y &= -(x-r)^2 + s. \end{aligned}$$

The vertices are at (p, q) and (r, s). The midpoint of these is

$$\left(\frac{p+r}{2}, \frac{q+s}{2}\right).$$

A rotation by 180° around this point transforms the one onto the other. □

3202. (a) Consider the $(n-k)$ lowest links as a single object. The mass of these is $(n-k)m$. Hence, the equation of motion is

$$\begin{aligned} & T_k - (n-k)mg = (n-k)ma \\ \Longrightarrow\ & T_k = (n-k)m(a+g). \end{aligned}$$

(b) Setting $a = 0$ for a hanging chain, we have $T_k = (n-k)mg$. This decreases linearly with k, i.e. down the chain. A heavy rope may be modelled as the mathematical limit of a heavy chain, as the links become very small. Hence, the same linear decrease in tension occurs in a rope as in a chain.

3203. There are ${}^8\text{C}_4 = 70$ equally likely outcomes. Of these, only two form squares (even vertices or odd vertices). Hence, the probability is $1/35$.

—— Alternative Method ——

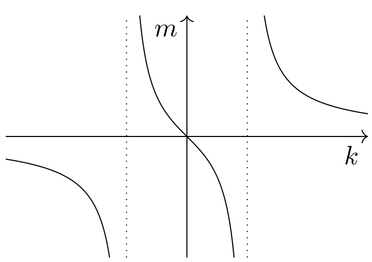
For a conditioning approach, the first vertex can be chosen wlog. For the second vertex there are three successful locations, for the third there are two, and for the last there is one:

$$p = 1 \times \tfrac{3}{7} \times \tfrac{2}{6} \times \tfrac{1}{5} = \tfrac{1}{35}.$$

3204. The gradients of the lines are

$$m = \frac{k}{k^2-1} \equiv \frac{k}{(k+1)(k-1)}.$$

Since this has two single asymptotes at $x = \pm 1$, and a single root in between them, its range is $\mathbb{R}$.



The only line left is $x = 0$. And setting $k = \pm 1$ (the values for which the gradient is undefined) in the original equation produces this line. Hence, this family contains all straight lines through the origin, as required.

3205. Writing in terms of 2^x, this is

$$\begin{aligned} & 8^x + 1 = 2^{x+1} \\ \Longrightarrow\ & (2^x)^3 - 2 \cdot 2^x + 1 = 0. \end{aligned}$$

This is a cubic in 2^x. It has $2^x = 1$ as a root, so $(2^x - 1)$ as a factor. Taking this out,

$$(2^x - 1)\big((2^x)^2 + 2^x - 1\big) = 0.$$

Using the quadratic formula,

$$2^x = 1, \tfrac{-1 \pm \sqrt{5}}{2}.$$

We reject the negative root, since $2^x > 0$, leaving $2^x = 1$ or $2^x = \frac{\sqrt{5}-1}{2}$. Taking logs, the former yields $x = 0$, the latter

$$\begin{aligned} x &= \log_2 \tfrac{\sqrt{5}-1}{2} \\ &= \log_2(\sqrt{5}-1) - \log_2 2 \\ &= \log_2(\sqrt{5}-1) - 1. \end{aligned}$$

Hence, the solution set is $\{0, \log_2(\sqrt{5}-1) - 1\}$.

3206. Differentiating with respect to y,

$$\frac{dx}{dy} = 64y^3 + 96y^2 + 40y + 8.$$

Hence, at $(0, -1/2)$, $dx/dy = 4$. So, the gradient of the tangent is $dy/dx = 1/4$. The equation of the tangent is $x = 4y + 2$. This meets the curve where

$$\begin{aligned} & 16y^4 + 32y^3 + 20y^2 + 8y + 2 = 4y + 2 \\ \Longrightarrow\ & 16y^4 + 32y^3 + 20y^2 + 4y = 0 \\ \Longrightarrow\ & y = -1, -1/2, 0. \end{aligned}$$

This gives $A : (-2, -1)$ and $C : (2, 0)$. These are equidistant from $B : (0, -1/2)$, as required.

3207. (a) Using $\sin 2x \equiv 2 \sin x \cos x$,

$$\begin{aligned} & \sin x \left(\tfrac{1}{2} \sin 2x - 1\right) \\ \equiv\ & \sin x(\sin x \cos x - 1) \\ \equiv\ & \sin^2 x \cos x - \sin x \\ \equiv\ & (1 - \cos^2 x) \cos x - \sin x \\ \equiv\ & \cos x - \sin x - \cos^3 x, \text{ as required.} \end{aligned}$$

(b) Using the above identity,

$$\begin{aligned} & \cos x = \sin x + \cos^3 x \\ \Longrightarrow\ & \cos x - \sin x - \cos^3 x = 0 \\ \Longrightarrow\ & \sin x \left(\tfrac{1}{2} \sin 2x - 1\right) = 0 \\ \Longrightarrow\ & \sin x = 0 \text{ or } \sin 2x = 2. \end{aligned}$$

The latter has no roots, as the range of the sine function is $[-1, 1]$, which leaves $x = 0, \pi$.

3208. The given solution may be expressed as $z = k_1x^3$ and $y = k_2x^{-1}$. Differentiating these,

$$\frac{dz}{dx} = 3k_1x^2 \quad \text{and} \quad \frac{dy}{dx} = -k_2x^{-2}.$$

Reciprocating the latter, we have $dx/dy = k_3x^2$. Since both dz/dx and dx/dy are proportional to x^2, they must be proportional to each other. Hence, $z \propto y^3 \propto x^{-1}$ satisfies the DE.

3209. The quotient must satisfy

$$\left(ax^3 + bx^2 + cx + d + \frac{e}{x+1}\right)(x+1) \equiv x^4 + 3x^2.$$

Equating coefficients, we require $a = 1$, then $b = -1$, $c = 4$, $d = -4$, $e = 4$. This is

$$\frac{x^4 + 3x^2}{x+1} \equiv x^3 - x^2 + 4x - 4 + \frac{4}{x+1}.$$

— Alternative Method —

Using polynomial long division,

$$\begin{array}{r|l} & x^3 - x^2 + 4x - 4 \\ \hline x+1 & x^4 + 3x^2 \\ & -x^4 - x^3 \\ \hline & -x^3 + 3x^2 \\ & x^3 + x^2 \\ \hline & 4x^2 \\ & -4x^2 - 4x \\ \hline & -4x \\ & 4x + 4 \\ \hline & 4 \end{array}$$

Rewriting this as an identity,

$$\frac{x^4 + 3x^2}{x+1} \equiv x^3 - x^2 + 4x - 4 + \frac{4}{x+1}.$$

3210. Solving simultaneously for intersections,

$$\begin{aligned} 8 - x &= \frac{27x}{(x+1)^2} \\ \Longrightarrow (8-x)(x+1)^2 &= 27x \\ \Longrightarrow x^3 - 6x^2 + 12x - 8 &= 0. \end{aligned}$$

A polynomial solver gives $x = 2$. Factorising, we get $(x-2)^3 = 0$. Hence, $x = 2$ is triple root, so the line is tangent to the curve.

3211. The derivative is $\frac{dy}{dx} = 9(3x+7)^2$. This is never negative, so the graph isn't decreasing anywhere. But the graph shown is clearly decreasing at $x = 0$. Hence, $y = (3x+7)^3 + 1$ could not generate the graph shown.

3212. (a) Using a calculator, $\mathbb{P}(L > 70) = 0.159$ (3sf).

(b) The probability that neither sablefish is over 70 cm in length is $(1 - 0.159)^2 = 0.707...$. So, the probability that at least one is over 70 cm is $1 - 0.707... = 0.292$ (3sf).

3213. (a) This is false: $a = b = \frac{1}{2}$ is a counterexample.

(b) This is false: $a = b = \sqrt{2}$ is a counterexample.

3214. A compound-angle formula gives

$$\sin(\alpha + \beta) \equiv \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Using radians, if β is small enough to ignore terms in β^2 and higher, then $\cos\beta \approx 1$ and $\sin\beta \approx \beta$. Substituting these in,

$$\sin(\alpha + \beta) \approx \sin\alpha + \beta\cos\alpha.$$

3215. Translating the entire problem by vector $b\mathbf{i}$, we look for the tangent to $y = x^3 + x^2 + x + 1$ at $x = 2$.

The derivative is $\frac{dy}{dx} = 3x^2 + 2x + 1$, which has value 17 at $(2, 15)$. The equation of the tangent is $y - 15 = 17(x-2)$, which is $y = 17x - 19$. Solving for intersections,

$$\begin{aligned} & x^3 + x^2 + x + 1 = 17x - 19 \\ \Longrightarrow\ & x^3 + x^2 - 16x + 20 = 0 \\ \Longrightarrow\ & x = -5, 2. \end{aligned}$$

Hence, the coordinates at which the tangent line re-intersects the curve are $(-5, -104)$.

The initial translation by $b\mathbf{i}$ has no effect on the y coordinate of the intersection, so the answer to the original problem is $y = -104$.

3216. Let $u = 1 + \sqrt{x}$. Then $du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$, which gives

$$dx = 2\sqrt{x}\,du = 2(u-1)\,du.$$

The limits are $u = 1, 2$. Enacting the substitution,

$$\begin{aligned} & \int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx \\ =\ & \int_1^2 \frac{u-1}{u} \cdot 2(u-1)\,du. \end{aligned}$$

Multiplying out and integrating, this is

$$\begin{aligned} & \int_1^2 2u - 4 + \frac{2}{u}\,du \\ =\ & \Big[u^2 - 4u + 2\ln|u|\Big]_1^2 \\ =\ & (-4 + 2\ln 2) - (-3 + \ln 1) \\ =\ & \ln 4 - 1, \text{ as required.} \end{aligned}$$

3217. There are 15 ways of choosing the chairperson, then ${}^{14}\mathrm{C}_2$ ways of choosing the secretaries, ${}^{12}\mathrm{C}_3$ ways of choosing the adjutants, and lastly ${}^{9}\mathrm{C}_4$ ways of choosing the liaisons. Overall, this gives

$$15 \times {}^{14}\mathrm{C}_2 \times {}^{12}\mathrm{C}_3 \times {}^{9}\mathrm{C}_4 = 37837800 \text{ ways}.$$

3218. Using a compound-angle formula, the LHS is

$$\begin{aligned}&\tan\left(\theta+\tfrac{\pi}{4}\right)\\ \equiv\ &\frac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}\\ \equiv\ &\frac{1+\tan\theta}{1-\tan\theta}.\end{aligned}$$

The RHS, using double-angle formulae, is

$$\begin{aligned}&\sec 2\theta+\tan 2\theta\\ \equiv\ &\frac{1}{\cos^2\theta-\sin^2\theta}+\frac{2\tan\theta}{1-\tan^2\theta}\\ \equiv\ &\frac{\frac{1}{\cos^2\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}+\frac{2\tan\theta}{1-\tan^2\theta}\\ \equiv\ &\frac{\sec^2\theta}{1-\tan^2\theta}+\frac{2\tan\theta}{1-\tan^2\theta}.\end{aligned}$$

Using the second Pythagorean identity, this is

$$\begin{aligned}&\frac{1+\tan^2\theta}{1-\tan^2\theta}+\frac{2\tan\theta}{1-\tan^2\theta}\\ \equiv\ &\frac{(1+\tan\theta)^2}{(1+\tan\theta)(1-\tan\theta)}\\ \equiv\ &\frac{1+\tan\theta}{1-\tan\theta}.\end{aligned}$$

This proves the identity.

3219. (a) Using $X \sim N\big(4.96, 0.02^2\big)$, the calculator gives $\mathbb{P}(X > 5) = 0.0228$ (3sf).

(b) With distribution $B(500, 0.02275)$, the mean is $np = 500 \times 0.0228 = 11.4$ (3sf).

(c) If $500p = 5$, then the probability of a failure needs to be $p = 0.01$. The inverse normal with $p = 0.99$, $\mu = 0$ and $\sigma = 0.02$ gives 0.046526.... Hence, the mean needs to be corrected to at or below $5 - 0.046526 = 4.9535$ mm (4dp).

(d) Using $p = 0.99$, $\mu = 0$ and $\sigma = 1$, we need $5 - 4.96 = 2.326347\sigma$. This gives $\sigma = 0.0172$ mm (4dp).

3220. The polynomial equation $\mathrm{f}(x) = \mathrm{g}(x)$ has degree at most n, so it can have at most n distinct roots. And we know that each of the n distinct reals in S satisfies it, since they produce $\mathrm{f}(x) = 0$ and $\mathrm{g}(x) = 0$, thus $\mathrm{f}(x) = \mathrm{g}(x)$. Hence, S is exactly the solution set of equation E:

(a) True,

(b) True,

(c) True.

3221. Let the digits in the first decimal place be $A < B$. So, we have

$$\begin{aligned}a &= 0.A[\text{other digits}]\\ b &= 0.B[\text{other digits}]\end{aligned}$$

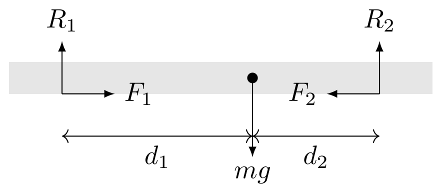
Consider $^B/_{10}$ (a.k.a. $0.B$ in the above notation). This is rational by construction. We now need to show that $^B/_{10}$ is in (a, b).

Since b is irrational, the decimal expansion of b cannot terminate; hence, its "other digits" part cannot be zero. This implies that $^B/_{10} < b$. Also, we know that $a < {}^B/_{10}$. So, $^B/_{10}$ is a rational number in (a, b), and we have proved the result by construction. □

3222. The sine function has period 360°. So, the series is periodic, with period 4. Furthermore, the unit circle (and thus sine graph) is symmetrical, so the sum of each set of four terms is zero. Hence, the sum of the series is also zero.

3223. The supports are identical, so we assume the same coefficient of friction μ at both.

Each support is attempting to move inwards, so the frictional forces on the rod act towards the centre of mass. The force diagram, therefore, is



We assume that the rod remains in equilibrium, while the supports move. Suppose $d_1 > d_2$. Then, taking moments around the centre of mass, $R_1 < R_2$, so the maximum frictional force is lower at support 1. Hence, if support 1 is further away, then it is support 1 which will slip.

This will continue until the two supports are equidistant from the centre of mass, at which point they will both slip equally. The supports will then meet at the centre of mass.

3224. (a) The obvious domain is the set of all polynomial functions.

(b) The set of non-negative integers $\mathbb{Z}^+$, or any set, such as $\mathbb{Z}$ or $\mathbb{R}$, which contains $\mathbb{Z}^+$.

(c) The equation $\mathrm{f}(x) = 0$ can have any number of distinct roots, so graphs of the form $y\,\mathrm{f}(x) = 1$ can have any number of asymptotes of the form $x = k$. The range, therefore, is the set of non-negative integers $\mathbb{Z}^+$.

3225. The formula for the sum of squares S_{xx} may be written in the following two equivalent forms:

$$\sum(x-\bar{x})^2 \equiv \Big(\sum x^2\Big) - n\bar{x}^2.$$

The LHS is non-negative, so $\sum x^2 \geq n\bar{x}^2$.

3226. Each term in the biquadratic has even degree. So, the inputs x and $-x$ produce the same y value. This means that a biquadratic graph of the form $y = ax^4 + bx^2 + c$ has even symmetry: the y axis is a line of symmetry.

A biquadratic is quartic, so it can have at most three stationary points. If two are minima, then the third, if it exists, can only be a maximum.

Furthermore, the minima must be reflections of one another in the y axis. Hence, they have the same y coordinate, and each must therefore be a global minimum of the curve. □

— Nota Bene —

If there are two minima, then the third stationary point must, in fact, exist. It must lie on the y axis. But this fact isn't necessary for the proof.

3227. (a) There are 4^4 ways of colouring the map, of which 4! are successful. So,

$$p = \frac{4!}{4^4} = \frac{3}{32}.$$

(b) The fact that the central region is blue gives no information as to the probability of there being exactly two colours. So, we can ignore it. The possibility space remains 4^4. Of these, there are ${}^4\text{C}_2 = 6$ ways of choosing two colours, and then $2^4 - 2$ ways of colouring the regions. So, the probability is

$$p = \frac{6 \times (2^4 - 2)}{4^4} = \frac{21}{64}.$$

— Alternative Method —

The central region is blue. This information restricts the possibility space to the 4^3 ways of colouring the rest.

Successful outcomes, classified by the total number of blue regions, are as follows:

① all outer regions are the same non-blue colour: 3 outcomes.

② one outer region is blue, and the other two are the same non-blue colour: $3 \times 3 = 9$ outcomes.

③ two outer regions are blue, and the other is non-blue: $3 \times 3 = 9$ outcomes.

Putting these together,

$$p = \frac{3 + 9 + 9}{4^3} = \frac{21}{64}.$$

3228. (a) $\text{f}'(x) = 270x^2 - 2242x + 3348.$

(b) The Newton-Raphson iteration is

$$x_{n+1} = x_n - \frac{90x_n^3 - 1121x_n^2 + 3348x_n - 2737}{270x_n^2 - 2242x_n + 3348}.$$

Starting at $x_0 = 1$, we get $x_1 = 1.30523$, then $x_2 = 1.39225$, and $x_n \to 1.4000$. This suggests $x = 7/5$ is a root, corresponding to $(5x - 7)$ as a factor.

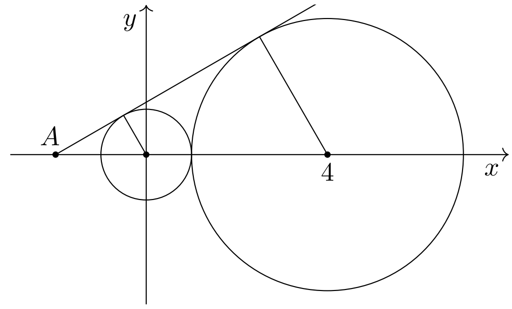
(c) We can verify the factor by taking it out:

$$\text{f}(x) = (5x - 7)(18x^2 - 199x + 391).$$

Factorising the quadratic gives

$$\text{f}(x) = (5x - 7)(9x - 23)(2x - 17).$$

3229. Both centres lie on the x axis; symmetry dictates, therefore, that L_1 and L_2 meet on the x axis. Hence, we only need consider the upper line L_1. The scenario is as follows.

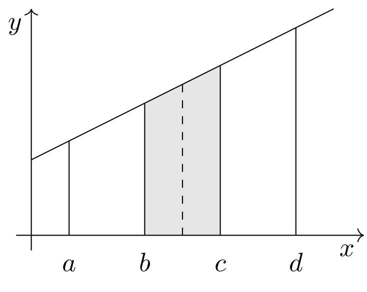


The right-angled triangles produced by L_1, the x axis and the radii are similar, with scale factor 3:

$$\frac{|OA| + 4}{|OA|} = 3$$
$$\Longrightarrow |OA| = 2.$$

Hence, A is $(-2, 0)$.

3230. Considered graphically, each integral represents the (signed) area under a straight line, i.e. the area of a trapezium:



Since the points $x = a, b, c, d$ are spaced at equal intervals, and f is linear, the average height (shown by the dashed line) of the two trapezia is the same. And the scale factor of the widths is 3, which proves the result.

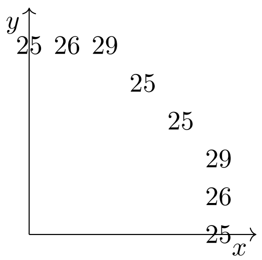
3231. All three have points of inflection.

(a) The second derivative is $6x$, which is zero and changes sign at $x = 0$.

(b) The second derivative is $12x^2-2$, which is zero and changes sign at $x = \pm 1/\sqrt{6}$.

(c) The second derivative is $20x^3 - 6x$, which is zero and changes sign at the origin (and also elsewhere).

3232. In $\mathbb{R}^2$, the region $20 < x^2 + y^2 < 30$ is an annulus or ring. In the +ve quadrant, the combinations of integer squares which add to give a value in $(20, 30)$ are shown below



Equivalent points appear in each of the quadrants. So, the total number of integer points (x, y) which satisfy $20 < x^2 + y^2 < 30$ is 4 on the axes, plus 4×6 not on the axes. This is 28 points in total.

3233. (a) The horizontal and vertical displacements are $x = pt$ and $y = qt - \frac{1}{2}gt^2$. Substituting the former into the latter,

$$\begin{aligned} y &= q\left(\frac{x}{p}\right) - \tfrac{1}{2}g\left(\frac{x}{p}\right)^2 \\ &\equiv \frac{q}{p}x - \frac{g}{2p^2}x^2. \end{aligned}$$

(b) Setting $y = 0$ gives $x = 0$ (launch) and $x = \frac{2pq}{g}$ (landing, thus range).

(c) Using the symmetry of a parabola, the angle below horizontal at which the projectile lands is the same as the angle above horizontal at which it is launched. This is $\theta = \arctan {}^{q}/_{p}$.

3234. The vertical distance between the curves is

$$\begin{aligned} &2 + \sin x + 3\cos x - (2\sin x + 4\cos x) \\ \equiv\ &2 - (\sin x + \cos x). \end{aligned}$$

When put into harmonic form, the expression in the brackets has amplitude $\sqrt{1^2 + 1^2} = \sqrt{2}$. Since this is less that 2, the vertical distance is never zero, and the curves do not intersect.

3235. The sum of a set of n standardised, independent normal variables has distribution

$$Z_1 + Z_2 + ... + Z_n \sim N(0, n).$$

So, in this instance, $Z_1 + Z_2 \sim N(0, 2)$. This gives $\mathbb{P}(Z_1 + Z_2 > 1) = 0.240$ (3sf).

Nota Bene

It is the result used in this question which allows for calculation of the distribution of sample means. Let $X \sim N(\mu, \sigma^2)$. Then

$$\bar{X} = \tfrac{1}{n}\sum_{i=1}^{n} X_i \equiv \sum_{i=1}^{n} \tfrac{1}{n}X_i.$$

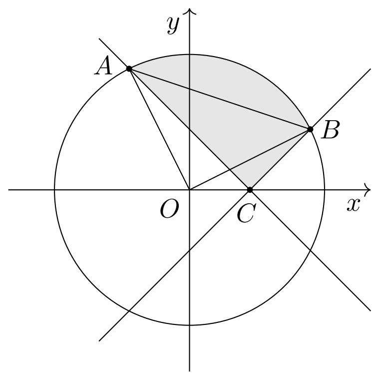
Each of these variables has distribution

$$\tfrac{1}{n}X_i \sim N\left(\frac{\mu}{n}, \frac{\sigma^2}{n^2}\right).$$

At this point we use the result in the question. Adding n instances of this variable multiplies both mean and variance by n. This scales the mean back up to μ and the variance to ${}^{\sigma^2}/_{n}$. Hence,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

3236. The region R in question, subdivided by chord AB into a segment and a triangle, is shaded below:



The vertices have coordinates

$$A : (-1, 2), \quad B : (2, 1), \quad C : (1, 0).$$

A and B are images of one another when rotated 90° around O, so the area of sector OAB is

$$A_{\text{sec}} = \tfrac{1}{4}\pi\left(\sqrt{5}\right)^2 = \tfrac{5}{4}\pi.$$

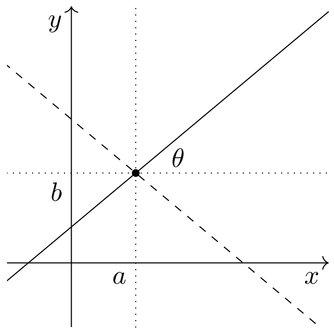
The area of $\triangle OAB$ is ${}^{5}/_{2}$. So, the area of minor segment AB is

$$A_{\text{seg}} = A_{\text{sec}} - A_{\triangle} = \tfrac{5}{4}\pi - \tfrac{5}{2}.$$

And the area of $\triangle ABC$ is 2. So, the area of R is

$$\begin{aligned} A_R &= A_{\text{seg}} + A_{ABC} \\ &= \tfrac{5}{4}\pi - \tfrac{5}{2} + 2 \\ &= \tfrac{5}{4}\pi - \tfrac{1}{2}. \end{aligned}$$

3237. By definition of tan, the angle of inclination of L is θ. Also, L passes through the point (a, b). Since both $x = a$ and $y = b$ pass through (a, b), so will the reflections. These are, in fact, the same line, shown dashed below:



The new angle of inclination for both is $-\theta$. Since $\tan(-\theta) \equiv -\tan\theta$, the equations are

(a) $y - b = -\tan\theta(x - a)$,

(b) $y - b = -\tan\theta(x - a)$.

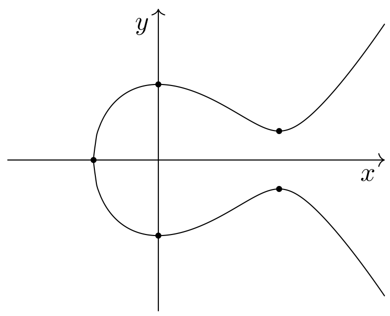
3238. (a) Close to the y axis, $|x|$ is small. Hence, x^3 is small compared to x^2, y^2 and 1. Hence, the curve approximates $x^2 + y^2 = 1$, which is a unit circle.

(b) Differentiating implicitly,

$$\begin{aligned} x^2 + y^2 &= \tfrac{5}{12}x^3 + 1 \\ \Longrightarrow 2x + 2y\frac{dy}{dx} &= \tfrac{5}{4}x^2 \\ \Longrightarrow \frac{dy}{dx} &= \frac{\frac{5}{4}x^2 - 2x}{2y} \\ &\equiv \frac{5x^2 - 8x}{8y}. \end{aligned}$$

This has roots where $5x^2 - 8x = 0$, which gives $x = 0, 1.6$. The gradient is undefined where $y = 0$. Hence, there are points with tangents parallel to the x axis at $(0, \pm 1)$ and $(1.6, \pm 0.383)$ and a point with tangent parallel to the y axis at $(-0.858, 0)$.

(c) As $x \to \pm\infty$, $y \to \pm\infty$. Combining this with the results of parts (a) and (b), the curve is



3239. Each factor is the same quadratic, in either x or x^3. Each has discriminant $\Delta = 69 > 0$. Hence, the quadratics have two roots each. So, since the cube root function is one-to-one, there are $2 + 2 = 4$ roots overall.

——— Nota Bene ———

We know that the factors <u>share</u> no roots, because the only fixed points of the cube root function are $x = \pm 1$, and these are not roots of $x^2 + x - 17$.

3240. The game must end eventually. So, if we look at the throw which ends the game, the possibility space is inner with probability 0.1 or outer with probability 0.2. Hence,

$$\mathbb{P}\,(\text{ends on inner bull}) = \frac{0.1}{0.1 + 0.2} = \frac{1}{3}.$$

3241. Expanding with compound-angle formulae,

$$\begin{aligned} &0 = 2\cos(\theta - \phi) - 2\cos(\theta + \phi) - 1 \\ \Longrightarrow\ &2(\cos\theta\cos\phi + \sin\theta\sin\phi) \\ &\quad - 2(\cos\theta\cos\phi - \sin\theta\sin\phi) = 1 \\ \Longrightarrow\ &4\sin\theta\sin\phi = 1. \end{aligned}$$

Substituting for $\sin\theta$,

$$\begin{aligned} &4\cos\phi\sin\phi = 1 \\ \Longrightarrow\ &2\sin 2\phi = 1 \\ \Longrightarrow\ &2\phi = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{13\pi}{6}, \tfrac{17\pi}{6} \\ \Longrightarrow\ &\phi = \tfrac{\pi}{12}, \tfrac{5\pi}{12}, \tfrac{13\pi}{12}, \tfrac{17\pi}{12}. \end{aligned}$$

3242. (a) This is false: $-1 > -2$, but $2 \not> 4$.

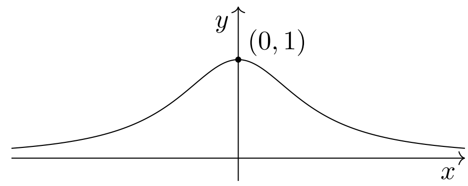
(b) This is true, since $x \mapsto e^x$ is increasing

(c) This is false: $\frac{1}{2} > 0$, but $-\frac{3}{8} \not> 0$.

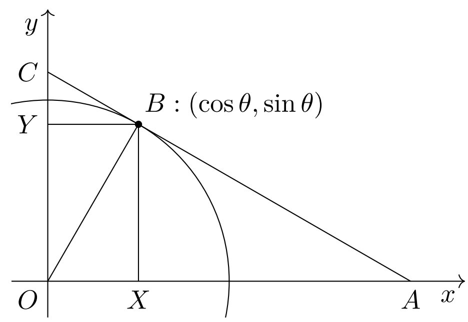
3243. Making y the subject, we have

$$y = \frac{1}{1 + x^2}.$$

Since x's index is even, this is symmetrical in the y axis (even symmetry), where it is maximised at $(0, 1)$. It has no other axis intercepts, and tends to 0 as $x \to \pm\infty$. So, the sketch is:



3244. The scenario is



Using similar triangles,

$$\begin{aligned}|OA| &= \sec\theta,\\ |OC| &= \operatorname{cosec}\theta,\\ |AB| &= \tan\theta,\\ |BC| &= \cot\theta.\end{aligned}$$

So, applying Pythagoras in the named right-angled triangles gives the following identities:

$$\begin{aligned}\triangle OAB &: \ 1+\tan^2\theta \equiv \sec^2\theta,\\ \triangle OCB &: \ \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta,\\ \triangle OAC &: \ \sec^2\theta + \operatorname{cosec}^2\theta \equiv (\cot\theta+\tan\theta)^2.\end{aligned}$$

3245. The squared distance between the asteroids is

$$\begin{aligned}S &= (5+t)^2 + (80-2t)^2 + (-10-2t)^2\\ &\equiv 9t^2 - 270t + 6525.\end{aligned}$$

This is minimised at $t = 15$, with $S = 4500$. So, closest approach is $\sqrt{4500} = 67.08 \approx 67$ metres.

3246. (a) By the quotient rule, the first derivative is

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}.$$

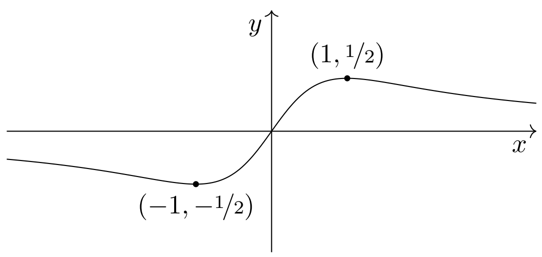
Setting the numerator to zero, we get $x = \pm 1$, so there are SPs at $(\pm 1, \pm 1/2)$.

(b) The second derivative is

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}.$$

The numerator has a single root at $x = 0$, so $f''(x)$ is zero and changes sign at the origin. So, the origin is a point of inflection.

(c) As $x \to \pm\infty$, $y \to 0^{\pm}$. Combining the above information (and the fact that the function has odd symmetry: rotational about the origin), the graph is

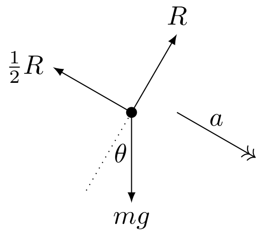


3247. Let $x = a^2 - 1$. Then the equation is

$$\begin{aligned}&x^3 - x = 0\\ \Longrightarrow\ &x = -1, 0, 1.\end{aligned}$$

So, $a^2 = 0, 1, 2$, giving $a = 0, \pm 1, \pm\sqrt{2}$.

3248. (a) The angle of inclination of the roof is given by $\theta = \arccos\frac{2.4}{3} = \arccos\frac{4}{5}$. The force diagram, with friction at $F_{\text{max}} = \frac{1}{2}R$ because the tile is in motion, is



(b) Perpendicular to the roof,

$$R = mg\cos\theta = \tfrac{4}{5}mg.$$

Parallel to the roof,

$$\begin{aligned}mg\sin\theta - \tfrac{1}{2}R &= ma\\ \Longrightarrow a &= \tfrac{3}{5}g - \tfrac{2}{5}g\\ &= \tfrac{1}{5}g.\end{aligned}$$

The speed is given by $v^2 = 2as$, so $v = \sqrt{1.2g}$. To 3sf, the horizontal and vertical components of speed are $v\cos\theta = 2.743428... = 2.74$ ms^{-1} and $v\sin\theta = 2.057575... = 2.06$ ms^{-1}.

(c) Upon leaving the roof, the tile can be modelled as a projectile. Vertically, with downwards taken as positive, we have $2.1 = 2.06t + 4.9t^2$. This gives a negative root, which we reject, and $t = 0.477541...$. Horizontally,

$$\begin{aligned}d &= 2.74 \times 0.477541...\\ &= 1.31 \text{ m (3sf)}.\end{aligned}$$

3249. The DE is satisfied if either $\frac{dy}{dx} = 1$ or $\frac{dy}{dx} = 2$. This is true for the first two graphs, but not for the third, whose first derivative is a non-constant linear function.

3250. Writing in harmonic form, we propose

$$\sin t + (\sqrt{2}-1)\cos t \equiv R\sin(t+\alpha).$$

Expanding, we need $R\cos\alpha = 1$, $R\sin\alpha = \sqrt{2}-1$. Using Pythagoras, and also dividing, these give

$$R = \sqrt{4-2\sqrt{2}}, \quad \alpha = \arctan(\sqrt{2}-1) = \tfrac{\pi}{8}.$$

Therefore,

$$\begin{aligned}&\sqrt{4-2\sqrt{2}}\sin(t+\tfrac{\pi}{8}) = 1\\ \Longrightarrow\ &\sin(t+\tfrac{\pi}{8}) = \frac{1}{\sqrt{4-2\sqrt{2}}}\\ \Longrightarrow\ &t+\tfrac{\pi}{8} = \tfrac{3\pi}{8}, \tfrac{5\pi}{8}\\ \Longrightarrow\ &t = \tfrac{\pi}{4}, \tfrac{\pi}{2}.\end{aligned}$$

3251. The range of $\sec x$ is $(-\infty, -1] \cup [1, \infty)$. This is then the domain of $x \mapsto e^x$, which is an increasing function. Under $x \mapsto e^x$, the constituent intervals of $(-\infty, -1] \cup [1, \infty)$ map as follows:

$$\begin{aligned}(-\infty, -1] &\longmapsto (0, 1/e]\\ [1, \infty] &\longmapsto [e, \infty).\end{aligned}$$

Hence, the range of $e^{\sec x}$ is $(0, 1/e] \cup [e, \infty)$.

3252. (a) By the chain rule, the first derivative is

$$\frac{dy}{dx} = \frac{2x}{x^2+1}.$$

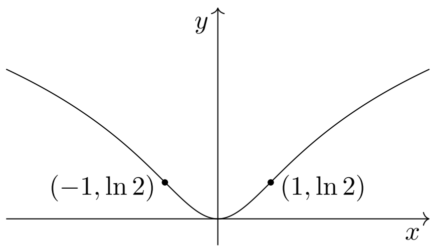
Setting the numerator of the derivative to zero, there is a stationary point at $(0, 0)$.

By the quotient rule, the second derivative is

$$\frac{d^2y}{dx^2} = \frac{2(1-x^2)}{(x^2+1)^2}.$$

The numerator has two single roots at $x = \pm 1$. A function changes sign at a single root, so there are points of inflection at $(\pm 1, \ln 2)$.

(b) The graph is defined over $\mathbb{R}$, and the range is $[0, \infty)$. The only axis intercept is O. As $x \to \pm\infty$, $y \to \infty$. Also, the graph is convex between the points of inflection, and concave outside them. Putting all of this together, the graph is



3253. (a) The translation is a weighted average of the vectors which would translate A to B, and A to C. So, the set of points attainable consists of all weighted averages of points B and C, i.e. the line BC.

(b) In terms of the individual position vectors,

$$\begin{aligned}\overrightarrow{AP} &= \mathbf{p} - \mathbf{a} \\ &= (\lambda_1 - 1)\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c}.\end{aligned}$$

So, we want

$$(\lambda_1 - 1)\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} = k_2(\mathbf{b} - \mathbf{a}) + k_3(\mathbf{c} - \mathbf{a}).$$

Equating coefficients of $\mathbf{b}$ and $\mathbf{c}$ gives $k_2 = \lambda_2$ and $k_3 = \lambda_3$. We can verify this by equating the coefficients of $\mathbf{a}$:

$$\lambda_1 - 1 = -\lambda_2 - \lambda_3,$$

which is exactly the condition that $\sum \lambda_i = 1$. So, we have

$$\overrightarrow{AP} = \lambda_2\overrightarrow{AB} + \lambda_3\overrightarrow{AC}.$$

(c) The result above tells us that $\overrightarrow{AP}$ is a weighted average (with $\lambda_2 + \lambda_3 < 1$) of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which run along the sides of $\triangle ABC$ to the other vertices. Since λ_2 and λ_3 are both positive, the vector $\overrightarrow{AP}$ (when placed at A), lies between sides AB and AC. And, since $\lambda_2 + \lambda_3 < 1$, the vector (when placed at A), stops short of side BC. Hence, point P lies inside triangle ABC. □

3254. Putting the fractions over a common denominator, we use two Pythagorean trig identities:

$$\begin{aligned}\mathrm{f}(x) &= \frac{1}{1+\sin x} + \frac{1}{1-\sin x} - 2 \\ &\equiv \frac{2}{1-\sin^2 x} - 2 \\ &\equiv \frac{2}{\cos^2 x} - 2 \\ &\equiv 2\sec^2 x - 2 \\ &\equiv 2\tan^2 x.\end{aligned}$$

3255. (a) We need $(a\sqrt{3} + b)^2 = 8 - 4\sqrt{3}$. Multiplying out and equating coefficients gives $3a^2 + b^2 = 8$ and $2ab = -4$. Solving these simultaneously, $a = \pm\sqrt{2}$ and $b = \mp\sqrt{2}$. So, the square roots are $\pm\big(\sqrt{6} - \sqrt{2}\big)$.

(b) Reciprocating the value for $\cot\phi$,

$$\begin{aligned}\tan\phi &= \frac{1}{2+\sqrt{3}} \\ &= 2 - \sqrt{3}.\end{aligned}$$

Then, using $\tan^2\phi + 1 \equiv \sec^2\phi$,

$$\begin{aligned}\sec^2\phi &= 1 + \big(2 - \sqrt{3}\big)^2 \\ &= 8 - 4\sqrt{3}.\end{aligned}$$

Hence, $\sec\phi = \pm\big(\sqrt{6} - \sqrt{2}\big)$. Since ϕ is acute, $\sec\phi$ must be positive, which means we take the positive root. Therefore $\sec\phi = \sqrt{6} - \sqrt{2}$, as required.

3256. Separating the variables,

$$\begin{aligned}&\frac{dy}{dx} = \frac{1-y}{1-x} \\ \Longrightarrow\ &\int \frac{1}{1-y}\,dy = \int \frac{1}{1-x}\,dx \\ \Longrightarrow\ &-\ln|1-y| = -\ln|1-x| + c \\ \Longrightarrow\ &1 - y = A(1-x).\end{aligned}$$

The point $(1, 1)$ renders both sides zero, regardless of the value of A. Hence, any solution curve must contain this point.

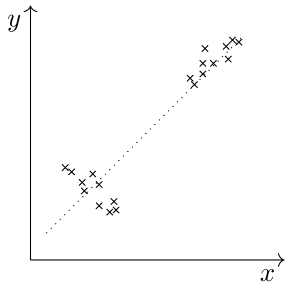
— Nota Bene —

The DE itself is not, in fact, defined at the point $(1, 1)$: substituting $(1, 1)$ anywhere before the last line in the working above would render both sides undefined. However, since a derivative cannot be defined at a single point, this doesn't render the DE meaningless: by writing it down in the first place we are assuming a limiting process which can cope with non-definition at a single point.

This is the reason for the word "polynomial" in the question. It rules out weird solutions such as

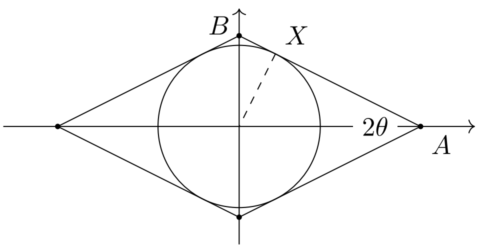
$$\{(x, y) \in \mathbb{R}^2 : x = y\} \setminus \{(1, 1)\}.$$

3257. A counterexample is as follows, in which the two samples, which have low spread, have means $(\bar{x}, \bar{y})$, which lie far apart on a line of positive gradient:



If the sample means lie far enough apart, relative to the spreads of the individual samples, then the correlation of the combined sample could be even stronger than that of either sample.

3258. The rhombus is as follows:



The direction of the dashed radius is $\pi/2 - \theta$, so the coordinates of X are $(\sin\theta, \cos\theta)$. Using similar triangles,

$$|AB| = |AX| + |XB| = \cot\theta + \tan\theta.$$

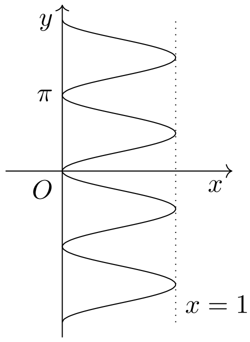
So, the perimeter is

$$\begin{aligned} P &= 4(\cot\theta + \tan\theta) \\ &\equiv \frac{4\cos\theta}{\sin\theta} + \frac{4\sin\theta}{\cos\theta} \\ &\equiv \frac{4(\cos^2\theta + \sin^2\theta)}{\sin\theta\cos\theta} \\ &\equiv \frac{4}{\frac{1}{2}\sin 2\theta} \\ &\equiv 8\,\text{cosec}\, 2\theta, \text{ as required.} \end{aligned}$$

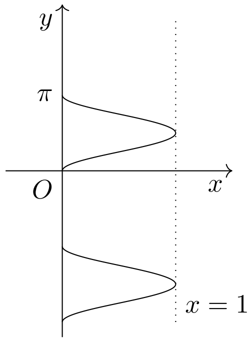
3259. (a) Using a double-angle formula,

$$\begin{aligned} x &= \sin^2 y \\ &\equiv \tfrac{1}{2}(1 - \cos 2y). \end{aligned}$$

This gives a cosine wave, bounded by $x = 0$ and $x = 1$, with period π in the y direction:



(b) This graph is a subset of that in (a). But any values for which $\sin y$ is negative, which were included in part (a), must now be removed:

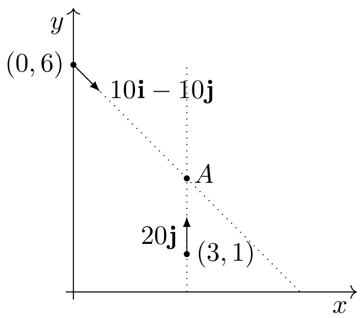


3260. Taking logs, base k, of the prime factorisation,

$$\begin{aligned} k &= p_1^{\alpha_1} \times p_2^{\alpha_2} \times \ldots \times p_n^{\alpha_n} \\ \Longrightarrow 1 &= \log_k \left(p_1^{\alpha_1} \times p_2^{\alpha_2} \times \ldots \times p_n^{\alpha_n}\right) \\ &\equiv \sum_{i=1}^{n} \log_k (p_i^{\alpha_i}) \\ &\equiv \sum_{i=1}^{n} \alpha_i \log_k p_i. \end{aligned}$$

So, the sum evaluates to 1.

3261. The first two forces are



Given equilibrium, the third force is $-10\mathbf{i} - 10\mathbf{j}$, with magnitude $10\sqrt{2}$ N.

And, since there is no resultant moment, we know that the lines of action of the forces are concurrent. The first two lines of action meet at $A : (3, 3)$. The third line of action, therefore, passes through this point, and so also through O, as required.

3262. Taking logs base 2 of the model equation,

$$\begin{aligned} & y = ax^n \\ \implies & \log_2 y = \log_2 a + n \log_2 x \\ \implies & \frac{\log_2 y - \log_2 a}{\log_2 x} = n \end{aligned}$$

This is constant, as required.

3263. The question suggests that there is only one value which lies in the range of both quadratics, over $\mathbb{N}$. Completing the square or differentiating gives

Sequence	Vertex
$a_n = 20n - n^2$	$a_{10} = 100$
$b_n = 500 - 40n + n^2$	$b_{20} = 100.$

And, since a_n is a positive quadratic and b_n is a negative quadratic, 100 is the only value in the range of both. So, $p = 10$ and $q = 20$.

— Nota Bene —

This is an example of a question which could be significantly harder if the numbers didn't work nicely. Such questions favour the bold: if you sit wondering about whether an approach will work, you'll think it won't. However, if you just pile in and see what happens, somehow it all works out!

3264. Since f is odd (no even powers), the graph $y = f(x)$ has rotational symmetry around the origin. And, since it is a polynomial, it must pass through the origin. So, all of these curves must intersect at the origin.

(a) Yes,

(b) Yes,

(c) Yes.

3265. Since the triangle is right-angled, the area is $\frac{1}{2}ab$. For this to be an integer, we need to show that at least one of a and b has a factor of 2. So, assume, for a contradiction, that a and b are both odd, with $a = 2p + 1$ and $b = 2q + 1$, where $p, q \in \mathbb{N}$. Then $a^2 + b^2 = c^2$ gives

$$\begin{aligned} (2p + 1)^2 + (2q + 1)^2 &= c^2 \\ 4(p^2 + p + q^2 + q) + 2 &= c^2. \end{aligned}$$

The LHS is even, so c must be even. Hence, c^2 must have a factor of 4. But the LHS doesn't have a factor of 4, since $4(p^2 + p + q^2 + q)$ does. This is a contradiction. Hence, at least one of a or b must be even, and the area must be an integer. QED.

3266. (a) The second derivative f″ is the gradient of the graph $y = f'(x)$. We require $f''(x) < 0$. The graph shown has a negative gradient for $x > -2$, so the set is $(-2, \infty)$.

(b) The first derivative is non-positive everywhere, so the cubic has no turning points (although it does have a stationary point of inflection at $x = -2$).

Hence, f is one-to-one. Since a cubic has range $\mathbb{R}$, the equation $f(x) = 0$ must therefore have exactly one root, as required.

3267. Solving simultaneously for vectors **a** and **b**, we get $\mathbf{a} = -2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$. Writing these as column vectors,

$$\mathbf{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

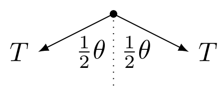
3268. (a) For three variables to change, we require the number 3 to be chosen, and then all variables to change:

$$p = \tfrac{1}{4} \times \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{32}.$$

(b) For two variables to change to 1, there are two cases. Either the number 2 is chosen, and both change, or the number 3 is chosen, and only two change:

$$p = \tfrac{1}{4} \times \left(\tfrac{1}{2}\right)^2 + \tfrac{1}{4} \times {}^3\mathrm{C}_2 \left(\tfrac{1}{2}\right)^3 = \tfrac{5}{32}.$$

3269. The force diagram is symmetrical about the angle bisector:



The forces cancel perpendicular to the dotted line. Along it, $F = 2T \cos \frac{1}{2}\theta$. We now use the double-angle formula $\cos^2 x \equiv \frac{1}{2}(\cos 2x + 1)$, with $x = \frac{1}{2}\theta$. We can take the positive square root, since θ is non-reflex, meaning that $\frac{1}{2}\theta$ is acute. This gives

$$\cos \tfrac{1}{2}\theta = \sqrt{1/2(1 + \cos\theta)}.$$

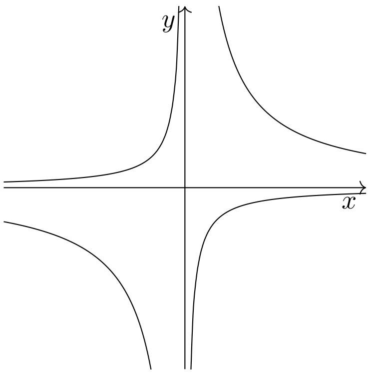
Substituting this in,

$$\begin{aligned} F &= 2T\sqrt{1/2(1 + \cos\theta)} \\ &\equiv T\sqrt{2(1 + \cos\theta)}, \text{ as required.} \end{aligned}$$

3270. The equation is a quadratic in xy:

$$\begin{aligned} & x^2y^2 - 10xy - 24 = 0 \\ \implies & (xy - 12)(xy + 2) = 0. \end{aligned}$$

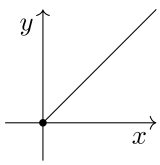
So, the hyperbolae are $xy = 12$ and $xy = -2$. Each occupies two diagonally opposite quadrants. The curve is:



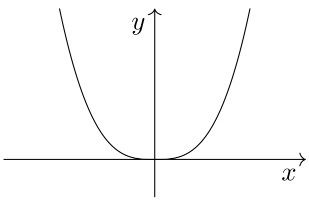
3271. (a) This is not true: $\mathrm{f}(x) = -x$, defined over $\mathbb{R}^+$, is a counterexample, since

$$\mathrm{f}(x) \wedge \mathrm{f}(x) = \sqrt{(-x)(-x)} = x.$$

(b) i. For positive x, this is $x \wedge x = \sqrt{x^2} = x$. For negative x, this is $x \wedge (-x)$, which is $\sqrt{-x^2}$. Since $-x^2 < 0$, this is undefined. The graph is



ii. For $x \geq 0$, we have $x^2 \wedge x^4 = \sqrt{x^6} = x^3$. For $x < 0$, we have $x^2 \wedge x^4 = \sqrt{x^6}$. This is a positive quantity, whereas x^3 is negative, so $y = -x^3$. Hence, the graph is $y = |x^3|$:



3272. Assume, for a contradiction, that k is the smallest positive irrational number. Then consider $\frac{1}{2}k$. If this was rational, then it would be $\frac{a}{b}$, so k would be $\frac{2a}{b}$, which would be rational. Hence, $\frac{1}{2}k$ is a positive irrational number which is smaller than k. This is a contradiction. Hence, there is no smallest positive irrational number. □

3273. Writing $\frac{u}{v} \equiv uv^{-1}$, we use product and chain rules:

$$\begin{aligned}\big(uv^{-1}\big)' &\equiv u'v^{-1} + u\big(v^{-1}\big)' \\ &\equiv u'v^{-1} - uv^{-2}v' \\ &\equiv \frac{u'v}{v^2} - \frac{uv'}{v^2} \\ &\equiv \frac{u'v - uv'}{v^2}, \text{ as required.}\end{aligned}$$

3274. The scale factor associated with an interest rate r is $(1 + r)$. So, at the end of the first month, the total outstanding is

$$T_1 = P(1 + r) - c.$$

We then scale this by $(1+r)$, and subtract another payment c, giving

$$\begin{aligned}T_2 &= \big(P(1 + r) - c\big)(1 + r) - c \\ &\equiv P(1 + r)^2 - c(1 + r) - c \\ &\equiv P(1 + r)^2 - c(2 + r), \text{ as required.}\end{aligned}$$

3275. Setting up the boundary equation and using a polynomial solver,

$$\begin{aligned}&Z^3 - Z - 1 = 0 \\ \Longrightarrow\ &Z = 1.3247.\end{aligned}$$

Since the cubic is positive, the successful region is $Z > 1.3247$. Using the normal distribution facility on a calculator, $\mathbb{P}(Z > 1.3247) = 0.0926$ (3sf).

3276. The first Pythagorean identity gives

$$\begin{aligned}\sin^2\theta &\equiv 1 - \cos^2\theta \\ &= 1 - \left(\frac{x^2 - y^2}{x^2 + y^2}\right)^2 \\ &\equiv \frac{(x^2 + y^2)^2 - (x^2 - y^2)^2}{(x^2 + y^2)^2} \\ &\equiv \frac{4x^2y^2}{(x^2 + y^2)^2}.\end{aligned}$$

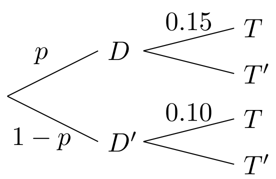
Since $\theta \in [0, \pi]$, $\sin\theta \geq 0$. So, we can take the positive square root, giving

$$\sin\theta = \frac{2xy}{x^2 + y^2}.$$

3277. The second square occupies the interval $y \in [0, 1]$. So, overlap could only occur while the first square has its lower left-hand corner at $y \in [-1, 1]$. This occurs for $t \in [-1, 1]$.

During this interval, $x_1 \in [-1, 1]$ and $x_2 \in [3, 5]$. Since these are separated by more than 1 unit of x distance, the squares never overlap.

3278. (a) The tree diagram is



We have been given $\mathbb{P}(D \mid T) = 0.3$. So,

$$\begin{aligned}&\frac{\mathbb{P}(D \cap T)}{\mathbb{P}(T)} = 0.3 \\ \Longrightarrow\ &\mathbb{P}(D \cap T) = 0.3\,\mathbb{P}(T) \\ \Longrightarrow\ &0.15p = 0.3\big(0.15p + 0.10(1 - p)\big) \\ \Longrightarrow\ &p = \tfrac{2}{9}.\end{aligned}$$

(b) The value 30% is not a known probability, but only a proportion in a sample. An inference from a sample to a population can only ever be an estimate.

3279. Factorising $e^{2x}-1$ as a difference of two squares,

$$\begin{aligned}&(e^{2x}-1)^3-(e^x-1)^3=0\\ \Longrightarrow\ &(e^x+1)^3(e^x-1)^3-(e^x-1)^3=0\\ \Longrightarrow\ &(e^x-1)^3\big((e^x+1)^3+1\big)=0.\end{aligned}$$

The second factor has no roots, because $(e^x+1)^3$ is positive. This leaves $e^x=1$. So, the solution is $x=0$.

3280. (a) Differentiating by the chain and product rules,

$$\begin{aligned}&y=e^{-\frac{1}{2}x^2}\\ \Longrightarrow\ &\frac{dy}{dx}=-xe^{-\frac{1}{2}x^2}\\ \Longrightarrow\ &\frac{d^2y}{dx^2}=-e^{-\frac{1}{2}x^2}+x^2e^{-\frac{1}{2}x^2}\\ &\qquad\;=(x^2-1)e^{-\frac{1}{2}x^2}.\end{aligned}$$

The values of the second derivative are:

x	0	$1/2$	1
$f''(x)$	-1	-0.662	0

(b) We can assume from (a) that the magnitude of the second derivative is larger for $0\le x\le 1/2$ than for $1/2\le x\le 1$. Where the magnitude of the second derivative is larger, the curvature is greater and the curve departs more from any line used to approximate it. Hence, errors will be proportionally greater for $0\le x\le 1/2$ than for $1/2\le x\le 1$.

3281. Consider $y=ax^2+bx+c$. We need to stretch this, scale factor -2, in the x direction. This gives

$$y=a\big(-\tfrac{1}{2}x\big)^2+b\big(-\tfrac{1}{2}x\big)+c.$$

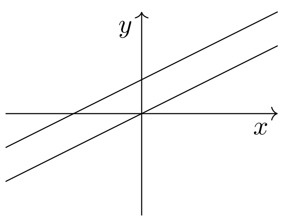
Simplifying, the relevant quadratic equation is

$$\tfrac{1}{4}ax^2-\tfrac{1}{2}bx+c=0.$$

3282. This is a quadratic in $(x-2y)$. Factorising,

$$\begin{aligned}&(x-2y)(x-2y+1)=0\\ \Longrightarrow\ &x-2y=0,-1.\end{aligned}$$

This gives a pair of parallel lines:



3283. Writing the given information algebraically,

$$\begin{aligned}\frac{3}{20}&=\frac{b}{b+r}\times\frac{b-1}{b+r-1},\\ \frac{1}{2}&=2\times\frac{b}{b+r}\times\frac{r}{b+r-1}.\end{aligned}$$

Multiplying up by the denominators,

$$\begin{aligned}3(b+r)(b+r-1)&=20b(b-1)\\ (b+r)(b+r-1)&=4br.\end{aligned}$$

Eliminating the pair of brackets $(b+r)(b+r-1)$ gives $12br=20b(b-1)$. We assume $b\neq 0$, so $r=5/3(b-1)$. Subbing into the second equation,

$$\begin{aligned}&\big(b+\tfrac{5}{3}(b-1)\big)\big(b+\tfrac{5}{3}(b-1)-1\big)=4b\cdot\tfrac{5}{3}(b-1)\\ \Longrightarrow\ &b=1,10.\end{aligned}$$

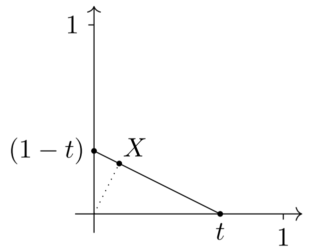
The former gives $r=0$, which we reject. So, the solution is $r=15$ and $b=10$.

3284. This is true. It is a general result in logic that the following two statements are identical:

① $X\implies Y$,
② $Y'\implies X'$.

Using this, we can reverse and negate $C'\implies B'$ giving $B\implies C$. The result is now obvious: the statements $A\implies B$ and $B\implies C$ are together sufficient to prove that $A\implies C$.

3285. (a) The line segment, including X for part (b), is



(b) The line PQ has gradient $\frac{t-1}{t}$ and equation

$$y-(1-t)=\tfrac{t-1}{t}x.$$

The normal to PQ through O has equation $y=\frac{t}{1-t}x$. Solving these simultaneously, X has coordinates

$$\left(\frac{t(1-t)^2}{2t^2-2t+1},\frac{t^2(1-t)}{2t^2-2t+1}\right).$$

(c) By Pythagoras,

$$\begin{aligned}|OX|&=\sqrt{\frac{t^2(1-t)^4}{(2t^2-2t+1)^2}+\frac{t^4(1-t)^2}{(2t^2-2t+1)^2}}\\ &\equiv\sqrt{\frac{t^2(1-t)^2(2t^2-2t+1)}{(2t^2-2t+1)^2}}\\ &\equiv\sqrt{\frac{t^2(1-t)^2}{2t^2-2t+1}}\\ &=\frac{t(1-t)}{\sqrt{2t^2-2t+1}}.\end{aligned}$$

(d) The original problem is symmetrical around $t = 1/2$, since t and $1 - t$ are equidistant from it. Hence, so must the expression for distance $|OX|$ be.

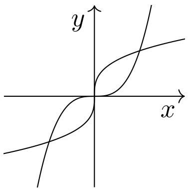
— Alternative Method —

Completing the square for top and bottom,

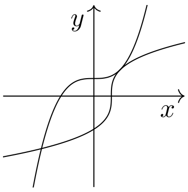
$$|OX| = \frac{-\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}{\sqrt{2\left(t - \frac{1}{2}\right)^2 + \frac{1}{2}}}.$$

This expression is symmetrical in $t = \frac{1}{2}$.

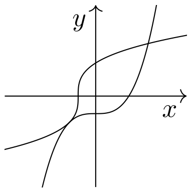
3286. This is true. Consider $p = 0$, which gives $y = x^3$ and $x = y^3$. The curves intersect three times:



As p increases from 0, the curves are translated, becoming tangent for one value $p > 0$. And the same occurs symmetrically as p decreases from zero. These are

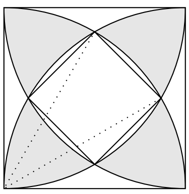


$p > 0$



$p < 0$

3287. The central region is a square and four segments. Since the intersections trisect the arcs, the angle subtended by these segments is $\frac{\pi}{6}$ radians.



Each (very thin) segment has area

$$\tfrac{1}{2} \cdot \tfrac{\pi}{6} - \tfrac{1}{2} \sin \tfrac{\pi}{6} = \tfrac{\pi}{12} - \tfrac{1}{4}.$$

The central square has side length

$$\sqrt{2\left(1/2 - \sqrt{3}/2\right)^2} = \sqrt{2 - \sqrt{3}}.$$

So, the central unshaded region has area

$$\begin{aligned} A_1 &= 4 \times \left(\tfrac{\pi}{12} - \tfrac{1}{4}\right) + \left(2 - \sqrt{3}\right) \\ &= \tfrac{\pi}{3} + 1 - \sqrt{3}. \end{aligned}$$

The ⬯-shaped region in common to two opposing sectors has area $A_2 = 2 \cdot \frac{\pi}{4} - 1 = \frac{\pi}{2} - 1$. So,

$$\begin{aligned} A_{\text{shaded}} &= 2A_2 - 2A_1 \\ &= 2\left(\tfrac{\pi}{2} - 1\right) - 2\left(\tfrac{\pi}{3} + 1 - \sqrt{3}\right) \\ &= \tfrac{\pi}{3} - 4 + 2\sqrt{3}, \text{ as required.} \end{aligned}$$

3288. Angle x, defined in radians, is small. Assuming that $3x$ and $4x$ are sufficiently small, $\sin 4x \approx 4x$ and $\cos 3x \approx 1 - \frac{1}{2}(3x)^2$. Substituting these in,

$$\begin{aligned} \frac{\cos 3x - 1}{x \sin 4x} &\approx \frac{1 - \frac{1}{2}(3x)^2 - 1}{x(4x)} \\ &\equiv \frac{-\frac{9}{2}x^2}{4x^2} \\ &\equiv -\tfrac{9}{8}, \text{ as required.} \end{aligned}$$

3289. The critical region has been split into two (2% has turned into 0.01), as if in a two-tailed test. But the form of H_1 dictates a one-tailed test. Otherwise, all is well. The final inequality should read

$$\mathbb{P}(X \geq c) < 0.02 < \mathbb{P}(X \geq c - 1).$$

3290. In both of the boundary cases, the line $3x+4y = k$ is tangent to the unit circle. The radii to the points of tangency have equation $y = -\frac{4}{3}x$. So, the points of tangency are $(\pm 3/5, \mp 4/5)$. These give $k = \pm 5$. To satisfy the original problem, we need k outside these boundary values:

$$k \in (-\infty, -5) \cup (5, \infty).$$

3291. (a) To reach the nth term, we take $n - 1$ steps, each of which is multiplication by the common ratio r. So, we multiply the first term by r^{n-1}, giving $u_n = ar^{n-1}$. □

(b) Calculating the quantity suggested, all but two terms cancel:

$$\begin{aligned} S_n - rS_n &= a + ar + ar^2 + ... + ar^{n-1} \\ &\qquad - ar - ar^2 - ... - ar^{n-1} - ar^n \\ &\equiv a - ar^n \\ &\equiv a(1 - r^n). \end{aligned}$$

Factorising, $S_n - rS_n$ is also equal to $(1-r)S_n$. This gives

$$\begin{aligned} &(1 - r)S_n = a(1 - r^n) \\ \Longrightarrow\ &S_n = \frac{a(1 - r^n)}{1 - r}, \text{ as required.} \end{aligned}$$

3292. Let $z = x^2 - 4$. This gives

$$\begin{aligned} &z^4 - z^3 = 0 \\ \Longrightarrow\ &z = 0, 1. \end{aligned}$$

Substituting x back in,

$$\begin{aligned} & x^2 - 4 = 0, 1 \\ \Longrightarrow\ & x^2 = 4, 5 \\ \Longrightarrow\ & x = \pm 2, \pm\sqrt{5}. \end{aligned}$$

3293. (a) Assuming that neither block moves, and that the friction between the blocks is limiting, horizontal equilibrium for the upper block gives

$$\begin{aligned} & 2g - \tfrac{2}{5}mg = 0 \\ \Longrightarrow\ & m = 5. \end{aligned}$$

So, the set of values for which the upper block moves is $[0, 5)$.

(b) Assuming that the upper block moves, and that the lower block is in limiting equilibrium, the friction between the blocks is at $\frac{2}{5}mg$, and the friction at the table is at

$$F_{\text{max}} = \tfrac{3+m}{5}g.$$

The lower block is on the point of moving if

$$\begin{aligned} & \tfrac{2}{5}mg - \tfrac{3+m}{5}g = 0 \\ \Longrightarrow\ & m = 3. \end{aligned}$$

So, the set of values for which the lower block doesn't move is $[0, 3]$.

(c) For $m \in [0, 5)$, the upper block moves. In this interval, the assumptions for (b) are met, so the set of values for which the lower block doesn't move is $[0, 3]$. Hence, the set of values for which all three blocks move is $m \in (3, 5)$.

3294. Let $u = \cos x$. This gives

$$\begin{aligned} & \frac{du}{dx} = -\sin x \\ \Longrightarrow\ & \frac{dx}{du} = -\frac{1}{\sin x}. \end{aligned}$$

By the chain rule,

$$\begin{aligned} \frac{d}{du}(\sin x) &= \cos x \cdot \frac{dx}{du} \\ &= \cos x \cdot -\frac{1}{\sin x} \\ &\equiv -\cot x, \text{ as required.} \end{aligned}$$

3295. (a) Taking out a factor of $(x-1)$,

$$\begin{aligned} & \lim_{x\to 1} \frac{x^2-1}{x^2-x} \\ =\ & \lim_{x\to 1} \frac{(x-1)(x+1)}{x(x-1)} \\ =\ & \lim_{x\to 1} \frac{x+1}{x} \\ =\ & 2. \end{aligned}$$

(b) Again taking out a factor of $(x-1)$,

$$\begin{aligned} & \lim_{x\to 1} \frac{x^3-1}{x^3-x} \\ =\ & \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x(x+1)(x-1)} \\ =\ & \lim_{x\to 1} \frac{x^2+x+1}{x(x+1)} \\ =\ & \tfrac{3}{2}. \end{aligned}$$

3296. (a) Using a calculator, $\mathbb{P}(X_1 > 50) = 0.159$ (3sf).

(b) The distribution of the sum of two such normal variables has distribution

$$X_1 + X_2 \sim N\left(80, 2 \times 10^2\right).$$

Using a calculator,

$$\mathbb{P}(X_1 + X_2 > 100) = 0.0786, \text{ (3sf)}.$$

3297. The line $x + y = k$ is perpendicular to $y = x$. So, to show that the third side of the triangle has the form $x + y = k$, we need only show that the first two lines are symmetrical in $y = x$, i.e. that their gradients are reciprocals:

$$\frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{4-3} = 2 - \sqrt{3}.$$

This proves the result.

3298. The boundary equation for the inequality is the parabola $y = x^2 - x$. Looking for intersections of the line and this parabola,

$$\begin{aligned} & \tfrac{1}{2}x - 1 = x^2 - x \\ \Longrightarrow\ & x^2 - \tfrac{3}{2}x + 1 = 0. \end{aligned}$$

This is a quadratic with $\Delta = -1.75 < 0$. Hence, the straight line doesn't intersect the boundary parabola. And, since the parabola is positive, we know that, for any point on the line $y = \frac{1}{2}x - 1$, $y < x^2 - x$. QED.

3299. (a) With side lengths $(2a, b, b)$, the perimeter gives $a + b = 32$ and the area gives $168 = a\sqrt{b^2 - a^2}$.

(b) Substituting to eliminate b,

$$\begin{aligned} & 84 = a\sqrt{(32-a)^2 - a^2} \\ \Longrightarrow\ & 168 = a\sqrt{1024 - 64a} \\ \Longrightarrow\ & a = \frac{168}{\sqrt{1024 - 64a}}. \end{aligned}$$

(c) Converting the above into an iteration,

$$a_{n+1} = \frac{168}{\sqrt{1024 - 64a_n}}.$$

Running this with $a_0 = 10$, we get $a_n \to 7$. This gives the side lengths of the triangle as $(14, 25, 25)$.

3300. Raising base and input to the same power, we can write $\log_4 x = \log_2 \sqrt{x}$. This gives

$$\begin{aligned} & 2^{\log_2 x} + 2^{\log_2 \sqrt{x}} = 12 \\ \Longrightarrow\ & x + \sqrt{x} - 12 = 0 \\ \Longrightarrow\ & (\sqrt{x} - 3)(\sqrt{x} + 4) = 0. \end{aligned}$$

The latter factor has no roots, as $\sqrt{x} > 0$, so $x = 9$.

——— End of 33rd Hundred ———